A New Accelerated Method for Solving Nonlinear Application Problems with Convergence of Order Two

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Abstract

The problem of locating roots of nonlinear equations occurs frequently in previous work. This study suggests a new accelerated iterative method for solving nonlinear equations. The new iterative method is converging quadratically. Some numerical problems illustrate that this new method can compete with Newton Raphson Method. Henceforth, it has been observed from the results and comparisons of developed method that the new accelerated Iterated Method is loftier than Newton Raphson Method.

Keywords: Taylor series, Adomian's method, Nonlinear equation; Newton Raphson method; Order of convergence

INTRODUCTION

The most significant and challenging problems is to find real roots of nonlinear equations, such as

$$f(x) = 0$$

There exist many applications that give thousands of the nonlinear problems. These applications rise in an extensive assortment of real-world applications in Applied Science and Engineering [1]-[3]. For this purpose, lots of numerical techniques had been suggested by using different techniques including quadrature formula, homotopy perturbation method and its variant forms, Taylor series, divided difference and decomposition method [4]–[10] Correspondingly, in this study, we have suggested an improvement of new iterated method by using [11, 12] and numerical condition for solving a real root of nonlinear equations. From results, it is shown that the recommended method converges quickly and is more competent with the assessment of Newton Raphson method. C++ programming is used to justify the results of proposed method. Numerical fallouts supports this theory as compared with Newton Raphson method for certain functions.

MATERIAL AND METHOD

Considering the nonlinear equation, such as

$$f(x) = 0 \tag{1}$$

Now writing $f(x + h)$ in Taylor's series expansion about x, we obtain

$$f(x+h) = f(x) + hf`(x) + g(h) \tag{2}$$
$$g(h) = f(x+h) - f(x) - hf`(x) \tag{3}$$

Supposing $f`(x) \neq 0$,for searching the value of h, so $f(x+h) = 0$, such that

$$f(x) + hf`(x) + g(h) = 0 \tag{4}$$

This is equivalent to finding the following `h`

$$h = -\frac{f(x)}{f`(x)} - \frac{g(h)}{f`(x)} \tag{5}$$

Eq (5) can be rewritten in the following form

$$h = c + N(h) \tag{6}$$

where

$$c = -\frac{f(x)}{f`(x)} \tag{7}$$

and

$$N(h) = -\frac{g(h)}{f`(x)} = \frac{-f(x+h) - f(x) - hf`(x)}{f`(x)} \tag{8}$$

Here c is a constant and N(h) is a nonlinear function. When we apply the technique of [11], in Eq (8) we get

$$S = \frac{-N(c+S^*)S + N(c+S)}{1 - N'(c+S^*)} \tag{9}$$

when x is sufficiently close to the real solution of f(x) = 0, $S^* \approx 0$. Thus Eq (9) is converted to

$$S = \frac{-N(c)S + N(c+S)}{1 - N'(c)} \tag{10}$$

Applying the Adomian's method to Eq (6), we get

$$A_0 = N(h_0) = N(c) = \frac{N(c)}{1 - N'(c)} = \frac{f(x+c)}{2f'(x) - 2f'(x+c)} \tag{11}$$

Now we construct the iterative method. For $h \approx h_0 = -\frac{f(x_n)}{f'(x_n)}$, obtains $h + x \approx x - \frac{f(x)}{f'(x)}$, which yields Newton method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

For k = 1, one obtains $h \approx h_0 + N(h_0), h + x \approx x + h_0 + N(h_0)$, which suggests the following iterative technique, we have

$$x = x_{n+1} - \frac{f(x_{n+1})}{2f'(x_n) - f'(x_{n+1})} \tag{12}$$

Eq (12) can also be written as

$$x = x_n - \frac{f(x_n)}{2f'(x_{n+1}) - f'(x_n)} \tag{13}$$

Where,

$$x_{n+1} = x_n + h \tag{14}$$

By using numerical condition i.e. $h = \Delta(x_n) = f(x_n)$ in Eq (14), then substitute in Eq (13), we get

$$x_{n+1} = x_n - \frac{f(x_n)}{2f'(x_n + f(x_n)) - f'(x_n)} \tag{15}$$

Hence Eq (15) is a new accelerated iterated method.

RATE OF CONVERGENCE

The following section shows that the New Accelerated Method is Quadratic Convergence.

proof

Using the relation $e_n = x_n - a$ in Taylor series, ore from Taylor series we estimate $f(x_n), f'(x_n)$ *and* $f'(x_n + f(x_n))$ with using this condition $c = \frac{f''(a)}{2f'(a)}$ and ignoring higher order term, we have

$$f(x_n) = f'(a)(e_n + ce_n^2) \tag{16}$$

or

$$f'(x_n) = f'(a)(1 + 2ce_n) \tag{17}$$

and

$$f\left(x_n + f(x_n)\right) = f`(a)[\left(e_n + f(x_n)\right) + c(e_n + f(x_n))^2$$

Or

$$f`\left(x_n + f(x_n)\right) = f`(a)[\left(1 + f`(x_n)\right) + 2c\left(1 + f`(x_n)\right)\left(e_n + f(x_n)\right)]$$

$$f`\left(x_n + f(x_n)\right) = f`(a)\left(1 + f`(x_n)\right)[1 + 2c\left(e_n + f(x_n)\right)] \tag{18}$$

By using Eq (16) and Eq (17) in Eq (18), we obtain

$$f`\left(x_n + f(x_n)\right) = f`(a)\left(1 + f`(a)(1 + 2ce_n)\right)[1 + 2ce_n\left(1 + f`(a)\right)]$$

$$f`\left(x_n + f(x_n)\right) = f`(a)\,[1 + f`(a) + 2ce_n + 6ce_n f`(a) + 2ce_n f`^2(a)] \tag{19}$$

By usingEq (19) and Eq (17), thus

$$2f`\left(x_n + f(x_n)\right) - f`(x_n) = f`(a)[2 + 2f`(a) + 12ce_n f`(a) + 4ce_n + 4ce_n f`^2(a) - 1 - 2ce_n]$$

$$2f`\left(x_n + f(x_n)\right) - f`(x_n) = f`(a)[[1 + 2f`(a) + 2ce_n\{1 + 6f`(a) + 2f`^2(a)\}] \tag{20}$$

Substituting Eq (16)*and* Eq (17)*in* Eq (15), we have

$$e_{n+1} = e_n - \frac{f`(a)(e_n + ce_n^2)}{f`(a)[[1 + 2f`(a) + 2ce_n\{1 + 4f`(a) + 2f`(a) + 2f`^2(a)\}]}$$

$$e_{n+1} = e_n - e_n(1 + ce_n)[1 + 2f`(a) + 2ce_n\{1 + 4f`(a) + 2f`(a) + 2f`^2(a)\}]^{-1}$$

$$e_{n+1} = e_n - e_n(1 + ce_n)[1 - 2f`(a) - 2ce_n\{1 + 4f`(a) + 2f`(a) + 2f`^2(a)\}]$$

$$e_{n+1} = 2e_n f`(a) + 2ce_n^2\left(\frac{3}{2} + 4f`(a) + 2f`^2(a)\right) \tag{21}$$

By using Eq (1) in Eq (16), then substitute *in* Eq (21), we get

$$e_{n+1} = \left[-2e_n^2 f``(a) + 2ce_n^2\left(\frac{3}{2} + 4f`(a) + 2f`^2(a)\right)\right]$$

$$e_{n+1} = e_n^2\left[-2f``(a) + 2c\left(\frac{3}{2} + 4f`(a) + 2f`^2(a)\right)\right]$$

Hence, this proves that the proposed iterative method has second order of convergence.

RESULTS AND DISSCUSSIONS

In this section, C++ programming is used to examine the fallouts of proposed method. The developed second order method is applied on few examples of nonlinear functions and interrelated with the Newton Raphson Method as shown in Table-1. From the numerical results, it has been observed that the second order accelerated method is reducing the number of iterations which is less than the iteration number of the Newton Raphson Method as well as accuracy as depicted in the following table.

Table-1: Numerical Results of New Method

FUNCTIONS	METHODS	ITERATIONS	ROOT	A E
e^{-x}-cosx	Newton Raphson Method	3	4.72129	$4.76837e^{-7}$
xo=5	New Method	2		$1.35655e^{-2}$
Sinx-x+1	Newton Raphson Method	8	1.93456	$1.19209e^{-7}$
xo=0.5	New Method	7		$1.27554e^{-5}$
$x^2 - e^x – 3x+2$	Newton Raphson Method	5	0.259171	$2.98023e^{-8}$
xo=1.5	New Method	4		$1.77413e^{-4}$
2x-lnx-7	Newton Raphson Method	4	4.21991	$4.76837e^{-7}$
xo=6	New Method	4		$4.76837e^{-7}$
sinx-0.5x	Newton Raphson Method	4	1.89549	$1.19209e^{-7}$
xo=2	New Method	4		$1.19209e^{-7}$

CONCLUSION

The problem of locating roots of nonlinear equations occurs frequently in scientific work. In this study, a new accelerated method is suggested and analyzed to determine the nonlinear problems. The developed method is convergence quadratically. Few examples exemplify the performance of the new method this makes it superior to the Newton iterative method with accuracy as well as iteration perception. Henceforth, it is observed from numerical outcomes that the proposed second order accelerated method is well execution, more effectual and informal to employment for solving non-linear equations.

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