On the Numerical Solution of Linear Multi-Term Fractional Order Differential Equations Using Laplace Transform and Quadrature

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Abstract

In this work, we extended the work of Sheen *et al.*, 2003 for the numerical solution of multi-term fractional order linear differential equations by an integral representation in the complex plane. The resultant integral is approximated to high order accuracy using quadrature. The accuracy of the method depends on the selection of optimal contour of integration. In the present work, linear multi-term fractional order differential equations are approximated for optimal contour of integration, and the results are compared with other methods available to demonstrate the accuracy and efficiency of the present numerical method.

Key words: Linear multi-term fractional order differential equations, Laplace Transform, Quadrature.

INTRODUCTION

In this work, we want to investigate the applicability of the proposed numerical scheme for the solution of linear multi-term differential equations of fractional order whose general form to be discussed is,

$$D^{\beta_1}u(t)+\upsilon_1 D^{\beta_2}u(t)+\upsilon_2 D^{\beta_3}u(t)+\upsilon_3 u(t)= f(t) \tag{1}$$

Where υ_i(i=1,2,3) and $0 < \beta_2 < \beta_3 < \beta_1$ are constants and the integer m is defined by $m \geq \beta_1 > m-1$, then one can specify suitable initial conditions,

$$u^i(0)=u_0^{\ i} \quad ,i=0,1,2,\ldots,m-1 \tag{2}$$

Where the Riemann-Liouville differential operators of fractional order $\beta > 0$, is defined as:

$$D^{\beta}u(t) = \frac{1}{\Gamma(m-\beta)}\frac{d^m}{dt^m}\int_0^t \frac{u(x)}{(t-x)^{\beta-m+1}}dx \tag{3}$$

Where the integer m is defined by m ≥ β > m-1(see [1], [2]). It is the problem described in (1) and (2) that shall address in the present paper. It has been shown that this problem has a unique solution using Laplace transform methods under some strong conditions (in particular, the linearity of the differential equations)[3]. In the last section of the paper, we discuss how the theoretical results may be applied in practical cases. In particular we consider the performance of existing numerical methods for solving linear multi-term differential equations of fractional order when the equations to be solved depend upon parameters that must be estimated and are subject to errors. We are aware of applications, from material sciences, for example, in which the order of the equation is a parameter estimated only to certain degree of accuracy. We investigated how to optimize quadrature step size for the present method in order to gain maximum accuracy with less computational cost.

Definition: If a (> -1) is a real number then:

$$L[t^a] = \frac{\Gamma(a+1)}{s^{a+1}} \tag{4}$$

WhereR(s) > 0

Lemma:

For m − 1 < β ≤ m, m ∈ N, the Laplace transformation of differential operator of fractional order can be written as [4]:

$$s^m F(s) - s^{m-1}f(0) - s^{m-2}f'(0) - \ldots - L\left[\left(\frac{d^{\beta}f}{dt^{\beta}}\right)(t)\right] = \frac{f^{m-1}(0)}{s^{m-\beta}} \tag{5}$$

METHOD

We select a quadrature step k > 0 and an equal weight quadrature formula in our numerical computations. The procedure is followed as applying Laplace Transformation to equation (1) for $\beta_1 \varepsilon (3, 4)$, $\beta_2 \varepsilon (0, 1)$ and $\beta_3 \varepsilon (1, 2)$, we get:

$$\hat{u}(s) = \left(s^{\beta_1} + v_1 s^{\beta_2} + v_2 s^{\beta_3} + v_3\right)^{-1} g(s) \tag{6}$$

Where

$$g(s) = s^{\beta_1 - 1}u_0 + s^{\beta_1 - 2}u_0' + s^{\beta_1 - 3}u_0'' + s^{\beta_1 - 4}u_0''' + \nu_1 s^{\beta_2 - 1}u_0 + \nu_2 s^{\beta_3 - 1}u_0 + \nu_2 s^{\beta_3 - 2}u_0' + \hat{f}(s) \tag{7}$$

And

$$\hat{f}(s) = L(f(t)) \tag{8}$$

Now using inverse Laplace transform to equation (6), we get

$$u(t) = \frac{1}{2\pi i}\int_{\Gamma} e^{st}\hat{u}(s)ds, R(s) \succ 0 \tag{9}$$

Where Γ is the integration contour such that $s_r \supset \Gamma$ and $s = s(v)$ given by equation (13). Now using equation (13) in equation (9), we get,

$$u(t) = \frac{1}{2\pi i}\int_{\Gamma} e^{s(v)t}\hat{u}(s(v))s'(v)dv \tag{10}$$

Now for the quadrature rule we choose the step k >0 and for simplicity we set $s_j = s(v_j)$, $s_j' = s'(v_j)$ where $v_j = kj$ for $N \geq j \geq -N$, we get

$$u_N(t) = \frac{k}{2\pi i}\sum_{j=-N}^{N} e^{s_j t}\hat{u}(s_j)s_j' \tag{11}$$

Now we solve the 2N+1 equations given in (12) for $N \geq j \geq -N$.

$$\hat{u}(s_j)\left(s_j^{\beta_1} + \nu_1 s_j^{\beta_2} + \nu_2 s_j^{\beta_3} + \nu_3\right) = g(s_j) \tag{12}$$

Contour of integration

We remarked that the numerical solution $\hat{u}(s_j)$ determined the approximate solution (11) for all 0 < t. However, in practice, the accuracy of approximate solution depends on the selection of the contour Γ. A number of such contour available one such path is due to [5] given as

$$s = \omega + \lambda(1 - \sin(\delta - iv)) \tag{13}$$

In fact, when $\operatorname{Im} v = \eta$, then (13) reduces to the left branch of hyperbola,

$$\left(\frac{x-\lambda-\omega}{\lambda\sin(\eta+\delta)}\right)^2-\left(\frac{y}{\lambda\cos(\eta+\delta)}\right)^2=1 \tag{14}$$

Here the strip $Y_r=\{v:|\operatorname{Im}v|\le r\}$

with r > 0 transformed into the hyperbola

$S_r=\{s:v\in Y_r\}\supset\Gamma$

Let $\sum_{\phi}=0\cup\{s\ne 0:|\arg s|\le\phi\}$

With $\pi/2(1-\alpha)\succ\phi\succ 0$

And let $\sum_{\beta}^{\omega}=\omega+\sum_{\beta}$, $\Gamma\subset S_r\subset\sum_{\beta}^{\omega}$.

NUMERICAL EXAMPLES

We test the present method for taking a number of problems to validate our numerical method for accuracy and efficiency. By comparing the results of the present method with other methods, it is clear that the present method is very convenient and effective.

Example No.1:

Here we apply the present numerical method to the equation [6], [7]

$$\begin{gathered}D^{\beta_1}u(t)+v_1D^{\beta_2}u(t)+v_2D^{\beta_3}u(t)+v_3u(t)=f(t)\\ u(0)=0,u'(0)=-1,u''(0)=2,u'''(0)=0\end{gathered} \tag{15}$$

Case1:

$$v_1=v_2=v_3=1,$$

$$\beta_1=3.91,\beta_2=0.77,\beta_3=1.44,$$

$$f(t)=\frac{2}{\Gamma(1.56)}t^{0.56}+\frac{2}{\Gamma(2.23)}t^{1.23}-\frac{1}{\Gamma(1.23)}t^{0.23}+t^2-t$$

Case2:

$$\nu_1 = \nu_3 = 0.5, \nu_2 = 1,$$

$$\beta_1 = \sqrt{11}, \beta_2 = \frac{\sqrt{2}}{20}, \beta_3 = \sqrt{2},$$

$$f(t) = \frac{2}{\Gamma(1.59)} t^{0.59} + \frac{1}{\Gamma(2.93)} t^{1.93} - \frac{0.5}{\Gamma(1.93)} t^{0.93} + 0.5(t^2 - t)$$

In such cases $u(t) = t^2 - t$ is the exact solution.

Table 1: Numerical results: at t=0.1, ω=2, θ=0.1, r=0.3431, δ=0.3812, and [t_0,T] =[0.01,1] corresponding to case 1 of (15).

N	Absolute error
5	0.0773
15	7.2720 e^{-005}
30	1.8211 e^{-007}
50	3.3186 e^{-010}
70	7.9374 e^{-013}
90	2.3740 e^{-015}
100	1.8709 e^{-016}
120	5.6477 e^{-017}

Table 2: Numerical results: at t=0.1, ω=2, θ=0.1, r=0.3431, δ=0.3812, and [t_0, T] =[0.01,1] corresponding to case 2 of (15).

N	Absolute error
5	0.0773
15	7.2720 e^{-005}
30	1.8211 e^{-007}
50	3.3186 e^{-010}
70	7.9368e^{-013}
90	2.2772e^{-015}
110	1.3886e^{-016}
120	1.2541e^{-016}

Table 1 and 2 shows the numerical results of the present method for case 1 and case 2 of example 1 respectively. The maximum absolute error for case 1 of example 1 obtained by[6] and [7] is 1.9e^{-006} and 9.53e^{-004} respectively, and the maximum absolute error for case 2 of example 1 obtained by[6] and [7] is 1.8e^{-007} and 9.80e^{-004} respectively. We have achieved better results than the results obtained by [6] and [7] of the same problem as shown in the table 1 and 2.

Example No.2:

Consider the equation [6] [8],

$$D^2 u(t) + D^{\frac{3}{4}} u(t) + u(t) = t^3 + 6t + \frac{8.533333333}{\Gamma(0.25)} t^{2.25} \tag{16}$$

Where

$u(0) = u'(0) = 0$

In such case $u(t) = t^3$ is the exact solution.

Table 3: Numerical results: at t=0.1, ω=2, θ=0.1, r=0.3431, δ=0.3812, and $[t_0,T]$ =[0.01,1] corresponding to (16).

N	Absolute error
5	0.1691
15	$8.3417\ e^{-005}$
30	$2.8584\ e^{-009}$
50	$1.6338\ e^{-014}$
70	$4.3748e^{-016}$
90	$1.8626e^{-016}$
100	$1.1279e^{-016}$
120	$4.1250e^{-017}$
158	$5.6722e^{-019}$

Table 3 shows the numerical results of the present method for example 2. The maximum absolute error for example 2 obtained by [6] and [8] is $3.39e^{-013}$ and $3.10e^{-006}$ respectively. We have achieved better results than the results obtained by[6] and [8] of the same problem as shown in the table 3.

CONCLUSION

From the results in the tables we have observed that the corresponding methodology is more efficient for approximating the solution of linear multi term differential equations of fractional order than other various methods. Therefore we finally conclude that by better selection of quadrature and contour we can improve the corresponding methodology in the future.

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