Numerical Solution of Fisher's Equation by Using Meshless Method of Lines

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Abstract

Many problems in science and engineering field are modeled by Partial Differential equations (PDEs). Non-linear, reaction diffusion equation, Fisher's equation, models different problems in ecology, biology and mass and heat transfer. This paper concerns with the development of meshless method of lines (MOL) for solving Fisher's equation. This method is applied in two steps. In the first step, Space derivatives are approximated by different radial basis functions. It results in conversion of PDE to system of ordinary differential equations (ODEs) which are then solved by Runge Kutta method of order 4 (RK4) in the second step. Finally L_2, L_∞ and root mean square (RMS) error norms are hired to check the behavior of the method. The proposed method is compared with some available methods in the literature.

Keywords: Multi Quadric (MQ), Inverse Multi Quadric (IMQ), Gaussian (GA), Inverse Quadric (IQ), Runge Kutta method of order 4 (RK4).

INTRODUCTION

In applied mathematics and physics, nonlinear phenomena play an essential role. For PDEs field of nonlinear control problems are the most mathematically challenging in the association of distributed parameter systems. Fisher's equation was first introduced by Fisher to model the advance of freak gene in an infinite domain and is nonlinear evolution equation [1]. Furthermore, Fisher's equation has been used as a basis to model the spatial spread of gene in chemical wave propagation, branching Brownian motion process, flame propagation, nuclear reactor theory [2-5], spread of invasive [6] , bacteria [7] , epidemics[8], and many other disciplines.

The Fisher's equation is defined by

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + \alpha u(1-u), \tag{1}$$

Where t shows the time and $x \in (-\infty, \infty)$ shows the position. A reactive coefficient α and diffusion coefficient D parameterized the reactive and diffusion process.

In this paper we have considered the following form of Fisher's equation,

$$u_t - u_{xx} - \alpha u(1-u) = 0. \tag{2}$$

Where α is an arbitrary constant.

The exact solution of Eq. (2) is given in [2],

$$u(x,t) = \frac{1}{\left[1 + e^{(\sqrt{\frac{\alpha}{6}})x - \frac{5\alpha.t}{6}}\right]^2}. \tag{3}$$

Many properties of Fisher's equation as a typical nonlinear reaction diffusion system have been studied by many authors, including the singular property and the travelling wave behaviour [2, 9-12]. Excellence summaries of Fisher's equation are provided by Kawahara & Tanaka [13], Baraznik & Tyson [14] and Larson [15]. The analytical study of Fisher's equation by using Adomian decomposition method is discussed by Wazwaz & Gorguis [2]. For the generalized Fisher's equation, the exact and explicit solitary wave solution has been presented by Wang [16]. The numerical solutions of Fisher equation were not present in the literature till 1947. First time Gazdag & Canosa [17], presented the numerical solution of Fisher's equation with a pseudo-spectral approach. After that Fisher's equation has been solved numerically by a lot of researchers. To discuss the numerical solution of Fisher equation, the implicit and explicit finite differences algorithms were presented by Twizell *et al.* [18] and Parekh & Puri [19]. Carey & Shen [20] used a least-squares finite element method. Comparison of nodal integral and non-standard finite schemes by Rizwan-Uddin [21] and Galerkin finite element method is proposed by Tang & Weber [22]. By using centered finite difference algorithm, the asymptotic boundary conditions are developed by Hagstrom and Keller [23]. A best finite-difference scheme for Fisher's equation proposed by Mickens [24], a pseudo spectral method was proposed by Olmos and Shizgal [25], a moving mesh method is used by Qiu & Sloan [26]. Also by applying wavelet Galerkin method, Fisher's equation is studied by Mittal *et al.* [27].

MOL is an efficient technique to find the numerical solution of PDEs. The main theme of this method is to discretize the space derivatives and the time derivatives remain continuous. The German mathematician Erich Rothe introduced MOL in 1930 [28]. He applied it to parabolic type equations, but it can be used in broad sense. MOL is considered as special case of FDM but it is more effective than FDM due to its accuracy and less computational cost.

We first discretize the given PDE in space variable by approximating the spatial derivatives by RBF/FDM and then solve the system of ODEs by any ODE solver. In recent years, different PDEs are solved by MOL, including, burger's type equation [29], generalized Kuramoto-Sivashinsky equation [30], KdV equation [31]. KdVequation for small time [32]. For stability and convergence of MOL see [33-35] .

FORMULATION OF METHOD OF LINES

In this section we will find the numerical solution of Equation (2) by applying MOL. By using RBF, the problem domain will be discretized in space variable which will convert the given PDE into system of ODEs, which will be easily solved by using any appropriate ODE solver. In this work we will use RK4 method.

RBF collocation

This section is concerned with MOL-RBF interpolation. By using RBF we will interpolate the approximate solution to the problem.

Let $\tilde{u}$ be the RBF approximation to u, which denote solution of the given PDE. Let us divide the problem domain in n nodes, $x_1, x_2, \cdots, x_n$, in $\Omega \cup \partial\Omega$ where Ω represents the interior and $\partial\Omega$ is the boundary of the domain. Also $x_1, x_n \in \partial\Omega$, while $x_2, x_3, \cdots, x_{n-1} \in \Omega$.

The RBF approximation of $u(x)$ is given by,

$$\tilde{u}(x) = \sum_{j=1}^{n} k_i \phi_j = k_1\phi_1 + k_2\phi_2 + \cdots + k_n\phi_n,$$

$$\Rightarrow \tilde{u}(x) = \Phi^T(x)\boldsymbol{k} \tag{4}$$

Where $\Phi(x) = [\phi_1(x), \phi_2(x), \cdots, \phi_n(x)]^T$ and $\boldsymbol{k} = [k_1, k_2, \cdots, k_n]^T$.

Here ϕ_j denotes the RBFs and k_j are the unknown constants.

Let we denote approximate solution is at j^{th} node by u_j i.e. $\tilde{u}(x_j) = u_j$ Then from Equation (4),

$$\begin{aligned} u_1 &= \Phi^T(x_1)\boldsymbol{k}, \\ u_2 &= \Phi^T(x_2)\boldsymbol{k}, \\ \vdots &= \vdots, \\ u_n &= \Phi^T(x_n)\boldsymbol{k}, \end{aligned}$$

Which can be written in the matrix form as,

$$\mathbf{B}\boldsymbol{k} = \mathbf{u}, \tag{5}$$

$$\text{Where } \mathbf{B} = \begin{bmatrix} \phi_1(x_1) & \phi_2(x_1) & \cdots & \phi_n(x_1) \\ \phi_1(x_2) & \phi_2(x_2) & \cdots & \phi_n(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_1(x_n) & \phi_2(x_n) & \cdots & \phi_n(x_n) \end{bmatrix},$$

$$\boldsymbol{k} = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} \text{ and } \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}.$$

In above equation matrix $\mathbf{B}$ is called the interpolation (or collocation) matrix and it consists of RBFs at nodes.
From Equation (5), we have,

$$\boldsymbol{k} = \mathbf{B}^{-1}\mathbf{u}$$

Putting this value in Equation (4), we get

$$\tilde{u}(x) = \Phi^T(x)\mathbf{B}^{-1}\mathbf{u},$$
$$\Rightarrow \tilde{u}(x) = \mathbf{P}(x)\mathbf{u}, \quad (6)$$

Here $\Phi(x) = [\phi_1(x), \phi_2(x), \cdots, \phi_n(x)]^T$, $\mathbf{u} = [u_1, u_2, \cdots, u_n]$ and $\mathbf{P}(x) = \Phi^T(x)\mathbf{B}^{-1}$,

The singularity of collocation matrix $\mathbf{B}$ depends on the choice of RBF. For this purpose; choose the value of shape parameter c as large as possible to get large difference between at least two columns of $\mathbf{B}$. However this produces less accuracy to contrast with that of small values of c. But small value of c causes ill-conditioning of collocation matrix. So the shape parameter has great effect on condition number [36]. The determination of best value of c is still problem. There are many methods to find the best value of c. The most easy and popular is the brute force method in which Max-error is plotted for different values of c. The value of c on which the least Max-error is appeared is considered as best value of shape parameter. This technique is applied to find the optimum value of c by the authors in [30] and [31]. Different methods of finding optimal value of c are given in [37-41].

Application of MOL-RBF to Fisher's Equation Using RBF

In this section, we will approximate the unknown solution $u(x)$ as a linear combination of n RBFs to find the numerical solution of Fisher's equation by applying MOL. We consider Fisher's equation,

$$u_t - u_{xx} - \alpha u(1-u) = 0 \quad (7)$$

Where α is arbitrary constant,
the initial condition is taken as,

$$u(x,0) = \psi(x) \; a \le x \le b, \quad (8)$$

and the boundary conditions are,

$$u(a,t) = \xi(t), \; u(b,t) = \eta(t). \quad (9)$$

Using Equation (6) to Equation (7), we get the following collocated form of Equation (7)

$$\frac{du_j}{dt} - \mathrm{P}_{xx}(x_j)u - \alpha u_j(1-u_j) = 0, \; j = 1,2,\cdots,n, \quad (10)$$

Where $u_j(t) = u_j$, and $\mathrm{P}_{xx}(x_j) = \left[\mathrm{P}_{1xx}(x_j) \quad \mathrm{P}_{2xx}(x_j) \quad \cdots \quad \mathrm{P}_{nxx}(x_j)\right]$,

With $P_{kxx}(x_j) = \dfrac{\partial^2}{\partial x^2} P_k(x_j)$.

To write the system of (10) as a column vector, let

$$U = [u_1 \quad u_2 \quad \cdots \quad u_n]^T,$$
$$P_{xx} = \left[P_{kxx}(x_j)\right]_{n\times n}.$$

Then Equation (10) can be written as,

$$\frac{dU}{dt} - P_{xx}U - \alpha U(1-U) = 0, \tag{11}$$

We can write Equation. (11) in compact form as,

$$\frac{dU}{dt} = F(U), \tag{12}$$

Where

$$F(U) = P_{xx}U + \alpha U(1-U). \tag{13}$$

The initial condition can also be written as,

$$U(t_0) = U_0 = \begin{bmatrix} u^0(x_1) & u^0(x_2) & \cdots & u^0(x_n) \end{bmatrix}^T. \tag{14}$$

From the boundary conditions,

$$u_1(t) = \xi(t),\ u_n(t) = \eta(t). \tag{15}$$

Now we will solve resulting system of ODEs (12) with initial conditions (14) by using RK4 method.

The RK4 algorithm for Equation (12) is

$$U^{m+1} = U^m + \frac{\Delta t}{6}(K_1 + 2K_2 + 2K_3 + K_4). \tag{16}$$

$$\text{Here } K_1 = F(U^m),$$

$$K_2 = F(U^m + \frac{\Delta t}{2}K_1),$$

$$K_3 = F(U^m + \frac{\Delta t}{2}K_2),$$

$$K_4 = F(U^m + \Delta t K_3).$$

Here U^m is the approximate solution at m^{th} time level, Δt is the time step.

RESULTS

In this section we will give numerical example to show the accuracy of our method. We will compare our method with DQM.

1.1 Test Problem 1

We consider the Fisher's equation,

$$u_t - u_{xx} - \alpha u(1-u) = 0 \tag{17}$$

Where α is an arbitrary constant.

The exact solution of Equation (17) is given by,

$$u(x,t) = \frac{1}{\left[1 + e^{(\sqrt{\frac{\alpha}{6}})x - \frac{5\alpha.t}{6}}\right]^2}, \tag{18}$$

The initial condition and boundary conditions are,

$$u(x,0)=\frac{1}{\left[1+e^{\left(\sqrt{\frac{\alpha}{6}}\right)x}\right]^2}, \tag{19}$$

$$u(a,t)=\frac{1}{\left[1+e^{\left(\sqrt{\frac{\alpha}{6}}\right)a-\frac{5\alpha.t}{6}}\right]^2}, \tag{20}$$

$$u(b,t)=\frac{1}{\left[1+e^{\left(\sqrt{\frac{\alpha}{6}}\right)b-\frac{5\alpha.t}{6}}\right]^2}. \tag{21}$$

Numerical calculation is performed over the interval $[0,1]\times[0,1]$ with step size $h=0.1$ and time step $\Delta t=0.001$, with $\alpha=6$. Different types of RBFs are used such as MQ, IMQ, GA and IQ. Table 2 is concerned with the error norms to assess the behavior of MOL, which are defined as follows:

$$L_\infty=\max\left|u^*-u\right|, \tag{22}$$

$$L_2=\sqrt{h\sum_{i=1}^{n}\left(u^*-u\right)^2}, \tag{23}$$

$$L_{rms}=\sqrt{\frac{\sum_{i=1}^{n}\left(u^*-u\right)^2}{n}}. \tag{24}$$

Where u^* and u are exact and numerical solutions respectively. From Table 1 it can be easily observed that MQ, IMQ, and GA have more accurate results than IQ. We have taken the values of $c=1,1.4,2.2$ and 0.34 for MQ, IMQ, GA and IQ respectively. In Table 2, there is comparison of MOL and DQM and it shows that MQ, IMQ and GA results are more accurate than that of DQM whereas IQ has less accuracy than DQM.

In this work, we have applied brute-force technique to find the best value of c. From Figure 1, we can see that the least Max-error for MQ is at $c=1.1$, Figure 2 shows that least Max-error for IMQ is at $c=1.4$. Similarly in Figure 3 one can see that least Max-error for GA occurs at $c=2.2$ and the Figure 4 represents that Max-error for IQ attains its minimum value at $c=0.34$.

Now we highlight the Figures of comparison of numerical and exact solution. In Figure 5, graph of numerical solution falls nearly to that of exact solution. And same as the

result for the graph of IMQ and GA, In Figure 6 and 7 respectively, but the graph of IQ in Figure 8 is slightly different from others.

Table 1: L_∞, L_2 and L_{rms} for different RBFs

t	RBF	L_∞	L_2	L_{rms}
0.1	MQ	2.6452E-07	1.3234E-07	1.3234E-07
	IMQ	3.5363E-07	2.1788E-07	2.1788E-07
	GA	7.5121E-07	2.8072E-07	2.8072E-07
	IQ	5.5756E-03	2.9324E-03	2.9324E-03
0.2	MQ	4.7441E-07	2.9961E-07	2.9961E-07
	IMQ	4.0007E-07	2.5327E-07	2.5327E-07
	GA	8.7103E-07	3.3578E-07	3.3578E-07
	IQ	2.2767E-02	1.1850E-02	1.1850E-02
0.3	MQ	7.3297E-07	4.9349E-07	4.9349E-07
	IMQ	4.6189E-07	2.4497E-07	2.4497E-07
	GA	9.3713E-07	3.6373E-07	3.6373E-07
	IQ	5.3162E-02	3.4769E-02	3.4769E-02
0.4	MQ	9.9249E-07	6.6783E-07	6.6783E-07
	IMQ	5.2276E-07	2.4321E-07	2.4321E-07
	GA	9.9070E-07	3.9942E-07	3.9942E-07
	IQ	8.8148E-02	6.0523E-02	6.0523E-02
0.5	MQ	1.1806E-06	7.9318E-07	7.9318E-07
	IMQ	6.1118E-07	2.6904E-07	2.6904E-07
	GA	1.0535E-06	4.4895E-07	4.4895E-07
	IQ	1.1332E-01	7.8289E-01	7.8289E-01

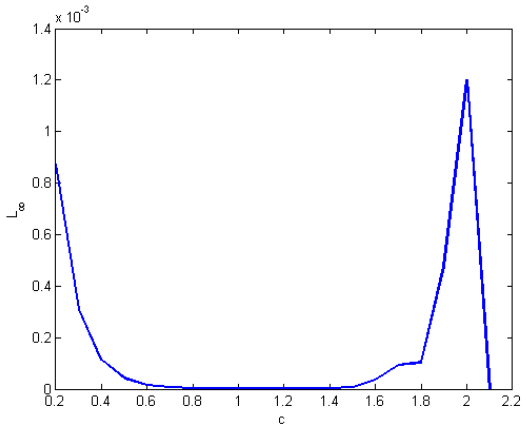

Figure 1: L_∞ error norm for different values of *c* using MQ

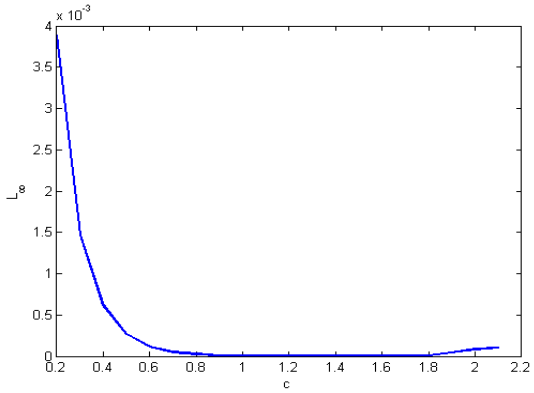

Figure 2: L_∞ error norm for different values of c using IMQ

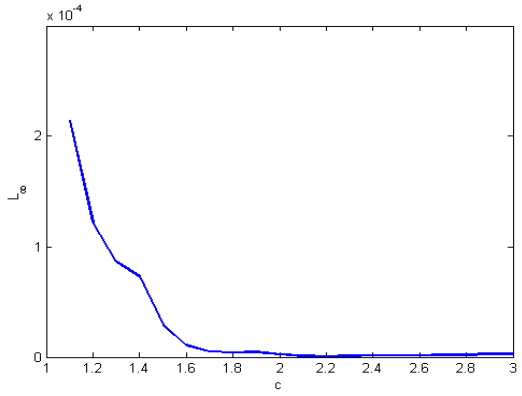

Figure 3: L_∞ error norm for different values of c using GA

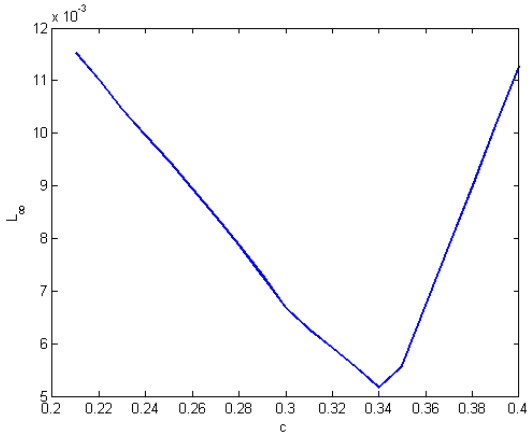

Figure 4: L_∞ error norm for different values of c using IQ

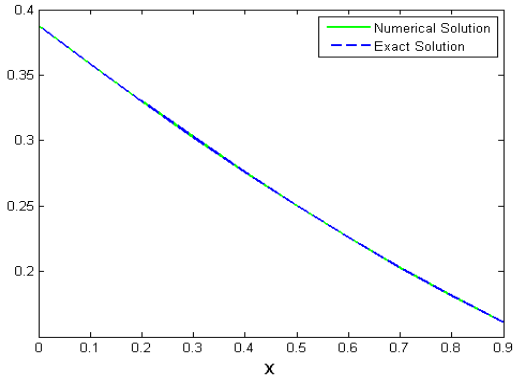

Figure 5: Comparison of numerical and exact solution using MQ

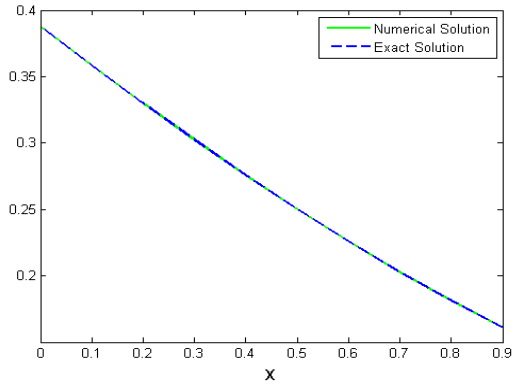

Figure 6: Comparison of numerical and exact solution using IMQ

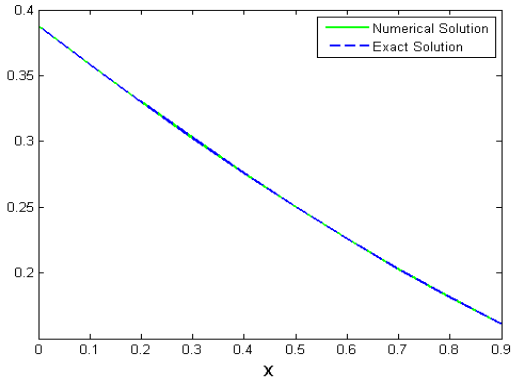

Figure 7: Comparison of numerical and exact solution using GA

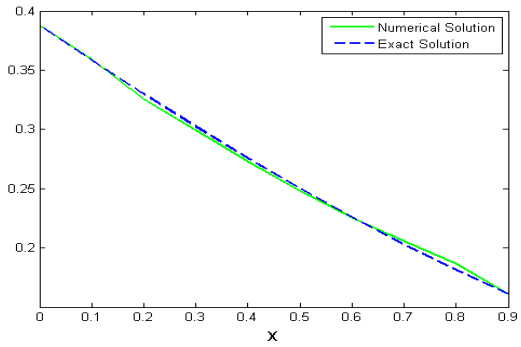

Figure 8: Comparison of numerical and exact solution using IQ

Table 2: Comparison of MOL RBF and DQM

x	t	MQ c=0.8	IMQ c=01	GA c=4.5	IQ c=0.01	Exact Solution	DQM [42]
0.25	0.5	0.818389	0.818393	0.818399	0.826779	0.818393	0.81843
	1.0	0.982915	0.982919	0.982926	0.999993	0.982919	0.98292
	2.0	0.999881	0.999883	0.999889	1.000312	0.999883	0.99988
	5.0	0.999999	0.999999	1.000004	1.000338	1.000000	1.00000
0.50	0.5	0.775800	0.775803	0.775811	0.769864	0.775803	0.77585
	1.0	0.978144	0.978147	0.978155	0.985615	0.978147	0.97815
	2.0	0.999849	0.999850	0.999857	1.000253	0.999850	0.99985
	5.0	0.999999	0.999999	1.000002	1.000290	1.000000	1.00000
0.75	0.5	0.725819	0.725823	0.725832	0.697997	0.725824	0.72588
	1.0	0.972068	0.972071	0.972080	0.979299	0.972071	0.92208
	2.0	0.999806	0.999808	0.999815	1.000333	0.999808	0.99981
	5.0	1.000000	0.999999	1.000000	1.000386	1.000000	1.00000

CONCLUSION

In this paper, we implemented MOL over Fisher's equation. Different types of RBFs are used to approximate the solution of the governing equation. Results show the impressive behavior of our method. We have compared our method with DQM and obtained more accurate results than DQM.

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