Numerical Solution of Initial Boundary Value Problems by MOL-RBF Method

FAZAL-I-HAQ[1], FARAH JAFFAR[2*], SIRAJ-UL-ISLAM[3],

[1] *Department of Mathematics, Stats and Computer Science, KPK Agricultural University, Peshawar, Pakistan*

[2] *Department of Mathematics, Shaheed Benazir Bhutto Women University, Peshawar, Pakistan*

[3] *Department of Basic Sciences, NWFP University of Engineering and Technology, Peshawar, Pakistan*

**Corresponding author's e-mail: farah.maths@gmail.com*

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Abstract

A recently developed meshless method, namely the radial basis function (RBF) combined with method of lines (MOL), i.e $\mathrm{MOL}-\mathrm{RBF}$ is used to find numerical solution of initial boundary value problems. Numerical examples are given to illustrate the practical usefulness of this approach. L_∞, L_2 norms and RMS are used for error estimation. Superiority of the proposed method is shown as compared with two existing numerical methods such as variational iteration method and finite difference method.

Keywords: Method of lines (MOL), Korteweg-de-Vries (KdV)equation, Radial basis function (RBF), Multiquadric (MQ), Inverse Multiquardic (IMQ), Guassian (GA), Fourthorder Runge Kutta (RK4), Finite difference method(FDM)

INTRODUCTION

In 1895, a partial differential equation was introduced by Korteweg and de Vries to model the height of surface of shallow water in the presence of long gravity waves which is called Korteweg-de-Vries (KdV) equation [1]. After 1960 Zabusky and Kruskal [2] numerically discovered the elastic collision between the KdV solitary waves and then Gardner *et al*.[3, 4]find the inverse scattering transform method and also solved the KdV equation analytically. That was a pioneering work which initiated many research activities on nonlinear waves. Since non linear phenomena plays crucial role in a variety of scientific fields, especially in fluid mechanics, solid state physics, thermodynamic etc [5].

Various researchers applied different numerical methods such as bilinear method of Hirota [6], the homogenous balance method [7], Sine-cosine method [8], Backlund

Transformation [9, 10], the Tanh method [11], the variational iteration method (VIM) [12, 13]Darboux transformation [14] etc to find the solution of nonlinear PDEs with appropriate initial or initial and boundary conditions.

We consider a nonlinear partial differential equation of the form;

$$\frac{\partial u}{\partial t}+\varepsilon u^m \frac{\partial u}{\partial x}+\mu \frac{\partial^3 u}{\partial x^3}=0, \qquad (1)$$

where ε and μ are constants and $m = 0,1,2.$

For $m = 0$ the Eq . (1) becomes linear KdV equation.

For $m = 1$ the Eq . (1) becomes nonlinear KdV equation [15].

For $m = 2$ the Eq . (1) becomes nonlinear Modified KdV equation [16].

KdV equation has some physical applications such as shallow water gravity waves, internal waves in the atmosphere and ocean etc. In fact it shows a combined effects of nonlinearity $(u\frac{\partial u}{\partial x})$ and the simplest longwave dispersion $(\frac{\partial^3 u}{\partial x^3})$ [4].

Since there is a class of nonlinear wave equations (soliton equations) which have the property of complete integrability such as Sine Gordon equation, the modified Korteweg-de-Vries equation and the cubic nonlinear Schrodinger equation [17].

We will study modify KdV equation.

$$\frac{\partial u}{\partial t}+\varepsilon u^2 \frac{\partial u}{\partial x}+\mu \frac{\partial^3 u}{\partial x^3}=0, \qquad (2)$$

we can also write this equation as

$$u_t+\varepsilon u^2 u_x+\mu u_{xxx}=0, \qquad (3)$$

mKdV equation describes the motion of waves in nonlinear optics, plasma or fluids [18]. It occurs in many nonlinear fields like acoustic waves in certain anharmonic lattices, schottky barriers of transmission lines, traffic congestion models, ion-acoustic solitons, Alfven waves in a collisionless plasma etc [7, 19]. It advocates several characteristic such as Conservation laws, N-solitons, Muira transformation, Inverse scattering transformation, Darboux transformation and Bilinear transformation [7].The modified KdV (mKdV) equation was solved by various numerical techniques both analytically and numerically such as Homotopy perturbation method (HPM) [19], Reduced differential transform method [5], Exp-function method [3], Backlund transformation [20], Adomian decomposition method (ADM) [15, 21]. The Extended Cosine-function is used to obtain the solution of traveling wave of the mKdV equation [22] , Homotopy analysis method

[18], Yuanxi [23] found series of explicit and exact solutions by Trial function method, Gesztesy *et al.*[24] applied Commutation method, Turabi and Dogan[16] used ADM and a 'lumped' Galerkin method with quadratic B-spline Finite Element method and obtained the numerical solution, Ibrahim and Kalaawy [22] used Extended Cosine-function to find the traveling wave solution of mKdV equation.

Meshless method is used in this study due to its remarkable properties of computing high dimension data, bringing changes in the domain of interest (like free surfaces and large deformations in the field of geometry) and consuming lesser time due to its independence from mesh. There are different meshless methods like Diffuse element method (DEM), Reproducing kernal particle method (RKPM), Moving Least square method, Radial basis function (RBF) [25].

The radial basis function (RBF) combined with method of lines (MOL), or $MOL-RBF$ is used to find numerical solution of initial boundary value problems i.e mKdV equations.

This paper is organized as follows.

In the introduction authors introduce the KdV and mKdV equations. Authors discuss RBF in detail that how they use the RBFs interpolation to approximate the solution in the section named Radial Basis Function. Method of lines for mKdV equation using RBFs is applied in next section. The section named Implementation and Results shows the results of numerical examples and comparison through tables and figures. After this convergence analysis is given and the last section includes conclusion.

Radial Basis Function

When a function to be approximated there are three cases to be considered

1. It depends on many variables or parameters,
2. It is defined by possibly many data,
3. The data are scattered in their domain.

The RBF approach is suitable for these cases [26].

It is one of the most advanced meshless method. It was first introduced in 1990 by Kansa[27] for the solution of PDEs. RBF has an edge over the other numerical methods like Finite volume method (FVM), Finite difference method (FDM) and Finite element method (FEM) because of its easy implementation, fast convergence [2], independent of mesh and accurate results.

Due to its properties, RBFs have been applied successfully to obtain numerical solution of many types of PDEs and ODEs including heat transfer equation, shallow water equation for tide and currents simulation, the nonlinear Burger equation upto two dimension [25]. Because of its meshless property it is useful for 3D problems as well as those problems that require re-meshing such as in nonlinear analysis [28]. It has many applications in science and mathematics such as mapping of two or three dimensional images like portraits or underwater sonar scan into other images for comparison [26].

RBF can be globally supporting, infinitely differentiable and consist of a shape parameter which needs to be selected from selected or random region for obtaining required accuracy [29].

Commonly available RBFs are

(1) **Multiquadric (MQ):** $\psi(r_j) = \sqrt{r_j^2 + c^2}$,

(2) **Inverse Multiquadric (IMQ):** $\psi(r_j) = \dfrac{1}{\sqrt{r_j^2 + c^2}}$,

(3) **Gaussian (GA):** $\psi(r_j) = e^{-c^2 r_j^2}$,

(4) **Quadric (Q):** $\psi(r_j) = (r_j^2 + c^2)$,

(5) **Inverse quadric (IQ):** $\psi(r_j) = \dfrac{1}{(r_j^2 + c^2)}$,

where c is the shape parameter.

Interpolation of RBF

Using $u^N(x)$ as the approximate function, RBF method gives [30]

$$u^N(x) = \sum_{k=1}^{N} c_k \psi(r)_k = \Psi^T(x)c, \tag{4}$$

where N is the number of data point, ψ is any form of RBF , $r_k = \|x - x_k\|$ which represents the Euclidean norm between collocation points x and x_k in the interval $[a,b]$

$\Psi(x) = [\psi_1(x), \psi_2(x) ... \psi_N(x)]^T$, $c = [c_1, c_2, ..., c_N]^T$,

Let $u^N(x_k) = u_k$, then

$$Ac = u, \quad \text{where} \quad u = [u_1, u_2, ..., u_N]^T \tag{5}$$

$$A = \begin{pmatrix} \Psi^T(x_1) \\ \Psi^T(x_2) \\ ... \\ \Psi^T(x_N) \end{pmatrix} = \begin{pmatrix} \psi_1(x_1) & \psi_2(x_1) & \cdots & \psi_N(x_1) \\ \psi_1(x_2) & \psi_2(x_2) & \cdots & \psi_N(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_1(x_N) & \psi_2(x_N) & \cdots & \psi_N(x_N) \end{pmatrix}$$

from Eq. (4) and Eq. (5), we get

$$u^N(x) = \Psi^T(x)A^{-1}u = M(x)u, \tag{6}$$

where $M(x) = \Psi^T(x)A^{-1} = [M_1(x), M_2(x), ..., M_{N-1}(x), M_N(x)]$,

Method of Lines for mKdV Equation Using RBFs

In this section we solve mKdV equation (3) in a finite domain

$$u_t + \varepsilon u^2 u_x + \mu u_{xxx} = 0, \tag{7}$$

$x \varepsilon [a, b]$

the initial condition is

$$u(x, t_0) = u^0(x) \tag{8}$$

and boundary conditions are

$$u(a,t) = f(t) \quad , u(b,t) = g(t) \tag{9}$$

here ε and μ are real constant and $u^0(x)$, $f(t)$, $g(t)$ are known functions.

Now we discretize the spatial derivatives according to the first step of MOL by using RBF interpolation . So that we choose N nodes in $[a,b]$

such as $a = x_1 < x_2 < \ldots < x_{N-1} < x_N = b$

by RBF interpolation the following results are obtained

$$u(x,t) \approx u^N(x,t) = \sum_{k=1}^{N} c_k \psi_k = \Psi^T(x)A^{-1}u = M(x)u \tag{10}$$

when we apply Eq. (10) to Eq. (7) , and collocate on the node x_k , we obtain

$$\frac{du_k}{dt} + \varepsilon u_k^2 (M_x(x_k) + \mu M_{xxx}(x_k) u = 0, k = 1,2 \ldots N \tag{11}$$

as $u_k(t)$ is abbreviated to u_k

as we have

$M_x(x_k) = [M_{1x}(x_k) M_{2x}(x_k) \ldots M_{(N-1)x}(x_k) M_{Nx}(x_k)]$

$$M_{lx}(x_k) = \frac{\partial}{\partial x} M_l(x_k) \quad , \quad l = 1,2\ldots, N$$

and also we have

$$M_{xxx}(x_k) = [M_{1xxx}(x_k) M_{2xxx}(x_k) \ldots M_{(N-1)xxx}(x_k) M_{Nxxx}(x_k)]$$

$$M_{lxxx}(x_k) = \frac{\partial^3}{\partial x^3} M_l(x_k), \quad l = 1,2,\ldots, N$$

We write this system in term of the column vectors.

Let $U = [u_1 u_2 \ldots u_{N-1} u_N]^T$,

$$\mathrm{M}_x = [M_{lx}(x_k)]_{N\times N} \quad ,$$

$$\mathrm{M}_{xxx} = [M_{lxxx}(x_k)]_{N\times N} \quad ,$$

So that above equation (11) can be written as

$$\frac{dU}{dt} + \varepsilon U^2 * (\mathrm{M}_x U) + \mu \mathrm{M}_{xxx} U = 0, \tag{12}$$

Where * is the multiplication of two vectors component-by-component.

We can also write the above equation as

$$\frac{dU}{dt} = F(U), \tag{13}$$

Where

$$F(U) = -\varepsilon U^2 * (\mathrm{M}_x U) - \mu \mathrm{M}_{xxx} U \quad ,$$

as the initial condition is

$$U(t_0) = [u^0(x_1) u^0(x_2) \ldots u^0(x_{N-1}) u^0(x_N)]^T, \tag{14}$$

and the boundary conditions are

$$u_1(t) = f(t), u_N(t) = g(t) \tag{15}$$

After completion of first step of $\mathrm{MOL-RBF}$ method we apply an ODE solver to solve equations . (13)-(15).

In this study fourth order Runge Kutta (RK4) is used as an ODE solver which is given by

$$U^{n+1} = U^n + \frac{\Delta t(K_1 + 2K_2 + 2K_3 + K_4)}{6} \quad ,$$

$$K_1 = F(U^n) \quad , \quad K_2 = F(U^n + \frac{\Delta t}{2} K_1) \quad ,$$

$$K_3 = F(U^n + \frac{\Delta t}{2} K_2) \quad , \quad K_4 = F(U^n + \Delta t K_3) \text{ [30]}$$

Implementation and Results

Example 1: Consider the mKdV equation along with the initial condition [15]

$$u_t + 6u^2 u_x + u_{xxx} = 0, \quad u(x,0) = a - \frac{4a}{4a^2 x^2 + 1}, \tag{16}$$

where "a" is a real constant , by taking $\varepsilon = 6$ and $\mu = 1$.
The exact solution is

$$u(x,t) = a - \frac{4a}{(4a^2(x - 6a^2 t)^2 + 1)}.$$

The boundary conditions are extracted from the exact solution.

The computational domain is $[0.1, 0.5]$, $t = 0.1 \ldots 0.5$, $\Delta t = 0.0001$, $N = 5, h = 0.1$.
We used the following error norms.

$$L_\infty = \left\| u^N - u \right\|_{max} = \max_{1 \le k \le N} | u_k^N - u_k |,$$

$$L_2 = \left\| u^N - u \right\|_{L_2} = \sqrt{h \sum_{k=1}^{N} (u_k^N - u_k)^2},$$

$$\text{Root mean square (RMS)} = \sqrt{\sum_{k=1}^{N} (u_k^N - u_k)^2 / N},$$

where u^N is approximate solution and u represents the exact solution.

In Table 1 RBFs shows higher accuracy than FDM as a result of use of L_∞, L_2 norms and Root mean square (RMS) for error estimation, which shows the accuracy of 10^{-5} and 10^{-6} by different values of shape parameter of RBFs while FDM shows 10^{-1} accuracy . The achievements of 5^{th} and 6^{th} order accuracy has been shown in Table 1. The Figures 1-2 show better graphical and 3D representation of approximate solution of RBFs with the exact solution of IBVP.

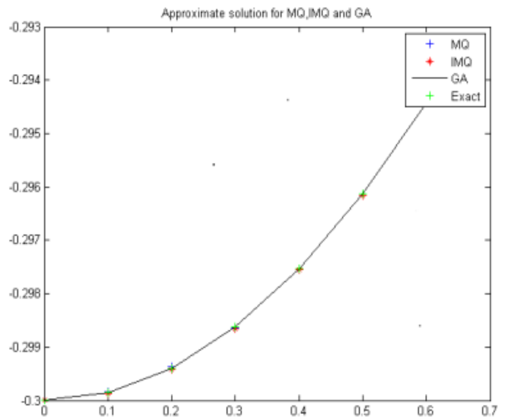

Figure 1: Approximate solution of RBFs of example 1

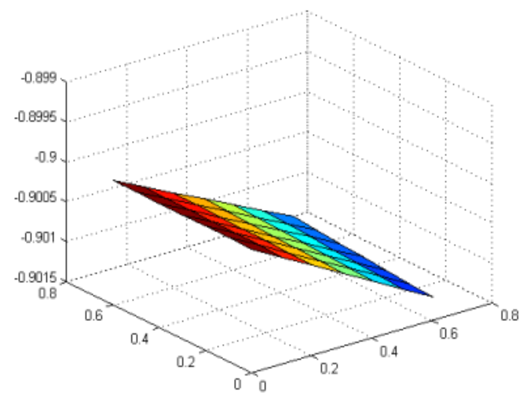

Figure 2: 3D graph of approximate solution of RBFs of example 1

Table 1: Computational domain is [0.1, 0.5] × [0.1, 0.5]

t	MQ		
	Max-error	L_2 **error**	**RMS**
0.1	2.2311E-5	1.0303E-5	1.2315E-5
0.2	4.8340E-5	2.2140E-5	2.6463E-5
0.3	7.5163E-5	3.4350E-5	4.1056E-5
0.4	1.0279E-4	4.6904E-5	5.6061E-5
0.5	1.3122E-4	5.9806E-5	7.1482E-5
	IMQ		
0.1	3.4265E-5	1.7730E-5	2.1192E-5
0.2	6.2099E-5	3.1012E-5	3.7066E-5
0.3	8.9720E-5	4.4179E-5	5.2804E-5
0.4	1.1711E-4	5.7212E-5	6.8382E-5
0.5	1.4425E-4	7.0102E-5	8.3788E-5
	GA		
0.1	2.4163E-6	1.1231E-6	1.3424E-6
0.2	4.8627E-6	2.2281E-6	2.6631E-6
0.3	7.3371E-6	3.3148E-6	3.9619E-6
0.4	9.8375E-6	4.3831E-6	5.2389E-6
0.5	1.2361E-5	5.4332E-6	6.4940E-6
	FDM		
0.1	1.0124E-1	5.1572E-2	6.1641E-2
0.2	2.2261E-1	1.0025E-1	1.1983E-1
0.3	3.6342E-1	1.5022E-1	1.7955E-1
0.4	5.0851E-1	2.0883E-1	2.4960E-1
0.5	6.3184E-1	2.7034E-1	3.2312E-1

Example 2: From the following mKdV equation the traveling wave solution was obtained [15].

$$u_t + 6u^2u_x + u_{xxx} = 0, \quad u(x,0) = \sqrt{b}\,\sec h(k + \sqrt{b}x), \tag{17}$$

for all $b \geq 0$, and k is an arbitrary constant. $N = 5, h = 0.1, \Delta t = 0.001$.

We have used RBFs such as MQ, IMQ, GA and the results are compared with FDM.

In Table 2 comparison shows that $\mathrm{MOL-RBF}$ has better results and higher order accuracy as compared to FDM. In the case of RBF, GA shows high accuracy than MQ and IMQ. The graphical results of approximate solution of RBFs and exact solution of IBVP are shown in Figure 3-4.

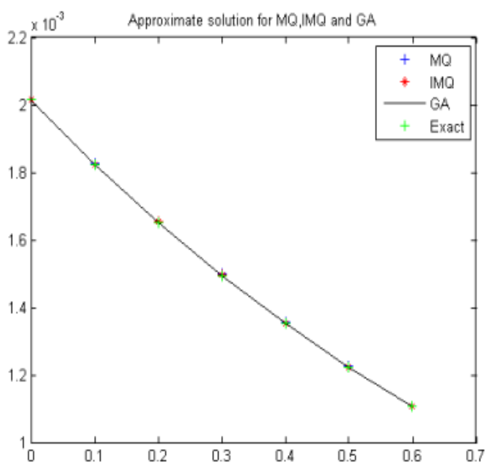

Figure 3: Approximate solution of RBFs of example 2

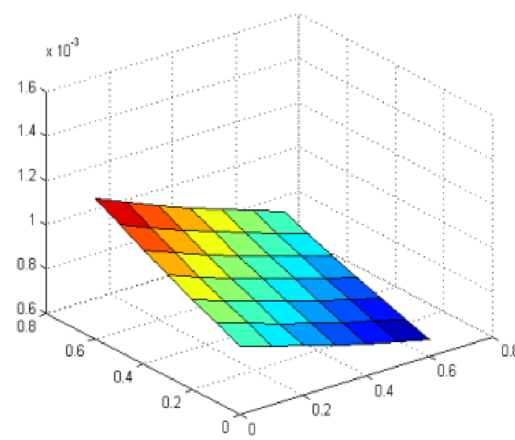

Figure 4: 3D graph of approximate solution of RBFs of example 2

Table 2: Numerical result for the traveling wave solution with $c = 1.0$, $k = 7$

t	Max-error	L_2 error	RMS
	MQ		
0.1	5.4013E-6	2.8995E-6	3.4655E-6
0.2	8.6746E-6	4.5079E-6	5.3879E-6
0.3	1.2434E-5	6.3992E-6	7.6485E-6
0.4	1.6751E-5	8.5572E-6	1.0227E-5
0.5	2.1484E-5	1.0969E-5	1.3110E-5
	IMQ		
0.1	3.2521E-6	1.5303E-6	1.8291E-6
0.2	1.0060E-5	5.2381E-6	6.2608E-6
0.3	1.7582E-5	9.4075E-6	1.1244E-5
0.4	2.5892E-5	1.4023E-5	1.6761E-5
0.5	3.5074E-5	1.9126E-5	2.286E-5
	GA		
0.1	2.5514E-8	1.6556E-8	1.9789E-8
0.2	6.0699E-8	3.4416E-8	4.1136E-8
0.3	1.0944E-7	5.3298E-8	6.3703E-8
0.4	1.6736E-7	7.2970E-8	8.7216E-8
0.5	2.2994E-7	9.3323E-8	1.1154E-8
	FDM		
0.1	1.4932E-2	7.4600E-3	8.9164E-3
0.2	3.0645E-2	1.5362E-2	1.8361E-2
0.3	4.8201E-2	2.4098E-2	2.8803E-2
0.4	6.7360E-2	3.3758E-2	4.0348E-2
0.5	8.8381E-2	4.4442E-2	5.3118E-2

Example 3: Consider mKdV equation [5]

$$u_t + 6u^2 u_x + u_{xxx} = 0, \tag{18}$$

with initial condition

$$u(x,0) = \sec h(x)$$

and exact solution given by

$$u(x,t) = \sec h(x-t), \tag{19}$$

In this example we have compared results with modified variational iteration method [31] and RBFs. The pointwise results are presented in Table 3 which demonstrates that at each point both MQ and GA produce better results. The Figure 5 shows graphical results of approximate solution of RBFs and VIM with exact solution and proved that RBFs are better. Figure 7 displayed pointwise absolute error of RBFs and VIM graphically, in which VIM shows varying error after nodal point 0.3 and at the point 0.8 the error shoot extensively.

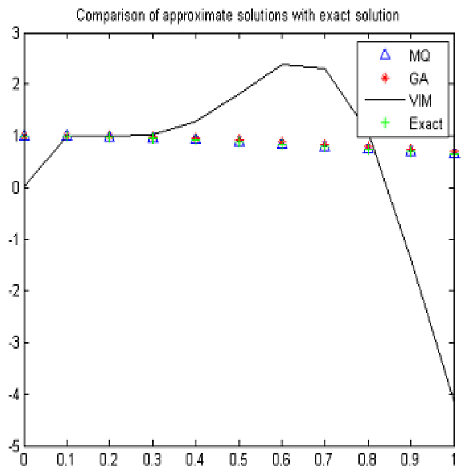

Figure 5: Approximate solution of RBFs and VIM of example 3

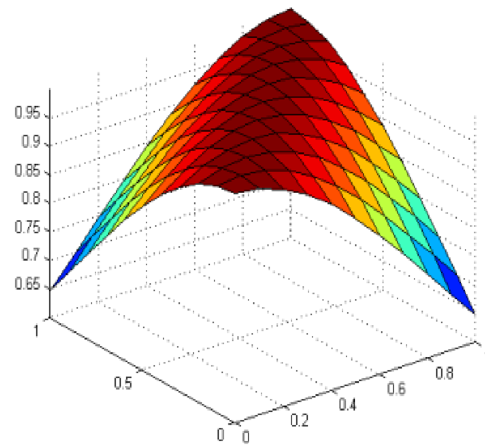

Figure 6: 3D graph of approximate solution of RBFs of example 3

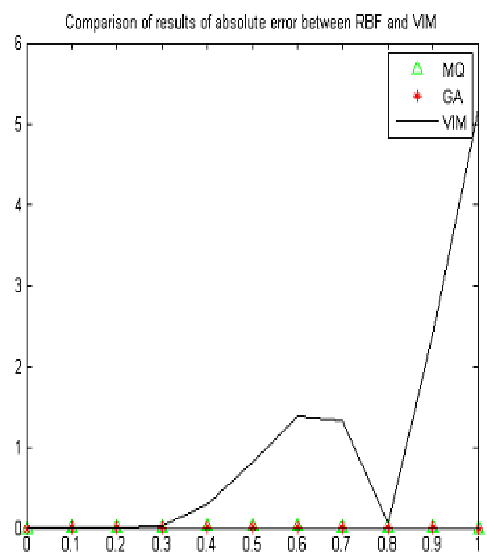

Figure 7: Comparative result of absolute error of RBFs and VIM of example 3

Table 3: Comparison of the absolute error of RBFs with VIM corresponding to example 3

x-t	MQ	GA	VIM
$0-0$	0	0	0
$0.1-0.1$	2.736E-4	1.649E-5	1.512E-3
$0.2-0.2$	4.879E-3	3.278E-5	9.902E-3
$0.3-0.3$	1.333E-2	4.815E-5	3.152E-2
$0.4-0.4$	2.342E-2	5.469E-5	2.818E-1
$0.5-0.5$	2.717E-2	1.767E-4	8.197E-1
$0.6-0.6$	2.363E-2	3.832E-4	1.3742
$0.7-0.7$	1.394E-2	1.311E-3	1.3184
$0.8-0.8$	3.273E-3	1.595E-3	6.357E-2
$0.9-0.9$	1.172E-4	1.638E-3	2.3936
$1.0-1.0$	1.110E-16	1.110E-16	5.1783

Convergence Analysis

The pointwise rate of convergence in space is computed by using the following formula

$$\frac{log_{10}(\|u-U_{hi}\|/\|u-U_{hi+1}\|)}{log_{10}(h_i/h_{i+1})} \quad [32]$$

In this formula u is the exact solution and U_{hi} is the numerical solution with space step size h_i. For the computation of rate of convergence in space for each MQ, IMQ and GA, the space step size h over the domain $[0.1,0.5]$ was kept fixed as $h=0.1$ while keeping the value of shape parameter constant at different collocation points N.

Table 4: L_∞ error norms and space rate of convergence at time t=0.1 of example 1

N_{times}	L_∞	order	L_∞	order	L_∞	order
		MQ		IMQ		GA
1000	2.2311E-5		3.4265E-5		2.4163E-6	
2000	1.8648E-5	0.25	2.5135E-5	0.44	6.6369E-8	5.18
4000	1.8110E-5	0.04	2.3320E-5	0.10	3.2585E-8	1.02
8000	1.7863E-5	0.01	2.2562E-5	0.04	1.5693E-8	1.05
16000	1.7744E-5	0.009	2.2215E-5	0.02	7.2465E-9	-0.48

Table 5: L_∞ error norms and space rate of convergence at time t = 0.1 of example 2

N_{times}	L_∞	order	L_∞	order	L_∞	order
		MQ		IMQ		GA
100	5.4013E-6		3.2521E-6		2.5514E-8	
200	9.3815E-7	2.52	1.3529E-6	1.26	3.6123E-7	-3.82
400	9.3811E-7	6.15	1.3333E-6	0.02	3.1026E-7	0.21
800	9.3809E-7	3.07	1.3273E-6	6.50	2.8619E-7	0.11
1600	9.3808E-7	0.000	1.3249E-6	2.61	2.8035E-7	0.02

In the Table 4 various collocation point N are used and the time step size $\Delta t = 0.0001$ had been kept constant. The result shows that rate of convergence in RBFs increases along with the improvement of accuracy L_∞ due to increase of collocation points N. In the Table 5 it is observed that the situation was same as it was noticed in Table 4, with fixed time step $\Delta t = 0.001$, but in the case of GA the accuracy reduced with the higher collocation points.

CONCLUSION

The different values obtained by the shape parameter which were selected randomly by proposed method gave the most accurate results. Significant differences are observed in the accuracy among RBFs at different shape parameters. It is also observed that the number of nodes and accuracy are inversely proportional to each other. The proposed method shows better numerical results for solving IBVPs. All the results indicate that Gaussian (GA) has the best accuracy as compared to MQ and IMQ of RBFs. The numerical results confirm that $\mathrm{MOL-RBF}$ method has better accuracy over the two numerical methods i.e FDM and VIM. The efficient results of numerical solution of nonlinear PDEs are obtained due to its two major advantages i.e the meshless property and use of the best ODE solver.

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